

## HOW TO FIND SPATIAL PERIODIC ORBITS AROUND THE MOON IN THE TBP<sup>\*</sup>

A. ARAM<sup>1, \*\*</sup>, H. ZOHOOR<sup>2</sup> AND S. SOHRABPOUR<sup>2</sup>

<sup>1</sup>School of Mechanical Engineering, Sharif University of Technology, Tehran, I. R. of Iran  
Email: aaram1980@yahoo.com

<sup>2</sup>Center of Excellence in Design, Robotics and Automation, Sharif University of Technology; Fellow,  
The Academy of Sciences of I. R. of Iran

**Abstract**– Finding special periodic orbit in the highly chaotic problems is very complicated. Thus, deriving an effective algorithm for finding such orbits is important. In this paper, the spatial orbits in three-body problems are studied. Then, an algorithm is derived in order to help to find these orbits. This algorithm includes scanning the space, refining better points, and finally, minimizing the position error of the orbit by means of numerical methods.

**Keywords**– Restricted three body problem, periodic orbits, algorithm, chaotic orbit

### 1. INTRODUCTION

The TBP (three body problem) is a well known problem in multi-body dynamics. In its traditional form, it was studied for the Sun- Earth- Moon system. Early works in TBP have been done by Euler. In the 18<sup>th</sup> and 19<sup>th</sup> century, many works originated from TBP. Some works about the application of TBP were done by Szebehely (1967) and Henon (1974). Some works in the literature are concerned with solving the TBP in special cases. For example, the Copenhagen problem ignores an important term in the equations of motion so that the equation becomes integrable and could be solved analytically. But the ignored term is essential and canceling it changes some important properties of the problem.

In the 60's and 70's more works were done for classification of orbits. The studies done by Szebehely and Henon had significant effects in those researches. That classification was according to the terms of equations described in the equations, or in restricted or non-restricted problem. In the restricted problem, the main assumption is that the third mass is ignorable and its motion does not influence two primaries Keplerian motion.

Hill's problem is another famous case in the TBP, in which it is assumed that the orbit of the third body is completely circular. Thus, the set of equations are changed into simpler forms representing a circular orbit for the two primaries. Since the orbits found in the circular and elliptical cases are similar to each other, the circular problem can be used as a base for the elliptical case, especially if the eccentricity of elliptical orbit for the two primaries is small. An interesting point in this problem is that the results of the circular case can be generalized in the elliptical case. Although this categorization of TBP reduced its generality in some cases, a major family of solutions in TBP was based on the type of the two primaries orbits.

The Spatial Restricted Three Body Problem (SRTBP) is another category, and can be described as the most general case, and the one most similar to the Earth-Moon-satellite problem. Regarding the

---

\*Received by the editors February 21, 2011; Accepted October 4, 2011.

\*\*Corresponding author

complexity of solving the SRTBP, it is rarely described. The improvements in recent space mission technologies have made the SRTBP more attractive and possible to be used for some space missions. Some researchers have studied the ability of using the spatial orbits of SRTBP as trajectories for space missions. Because of the instability and complexity of these orbits, less work has been done on the SRTBP. Of course, the advantages of these orbits studied in this paper demonstrate their capability to be used in space mission designs.

Early studies in the applications of TBP focus on settling a space telescope or a satellite in the stable Lagrange points in the Earth-Moon-satellite or the Sun-Earth-satellite problem. Although the satellite orbit may change due to some unpredictable forces and should be stabilized by applying control actions, the intrinsic stability of orbit lowers the amount of required control efforts.

Another major application of TBP is studying the chaos in the problem. Attractors of chaotic systems could be used as a basis for the space mission designs and the passive satellite idea seems to be practical.

Henon (1974) and Szebehely (Szebehely and Nacozy (1967); Szebehely (1971, 1974), Canalias and Masdemont (2008), Boltt and Meiss (1995) and Gomez et al. (2004) have some major works about planar RTBP. However, the works performed on studying the spatial cases are not so significant. In the spatial branch, Palacian et al. (2006) made an effort to find the periodic orbits in general form. But no periodic orbits were found and only formulations that prove their existence were presented. In their works, by using two successive integrations, the Hamiltonian system is reduced to a new system without time in the equations, and stability of periodic orbits was studied. No study on the chaos measurement has been done. Aram et al. (2010) presented a new algorithm to find spatial orbits in the three-body problem. They also studied chaos in the obtained orbits.

## 2. THREE-BODY PROBLEM

The main problem studied in this research is the simplified three-body problem in which the two primaries have elliptical orbit and the third body's mass is considered small enough so that it does not affect the motion of the two primaries. Thus, the most similar system to this one is the Earth-Moon-satellite or the Sun-Earth-satellite system.

As mentioned above, the third body does not affect the motion of the two primaries, because the mass ratio of the third body is in order of  $10^{-22}$  or less. Using this assumption, the equations of motion are:

$$\ddot{X} = \frac{-(X - X_e)}{\left((X - X_e)^2 + (Y - Y_e)^2 + Z^2\right)^{\frac{3}{2}}} + \frac{-(X - X_m)}{\left((X - X_m)^2 + (Y - Y_m)^2 + Z^2\right)^{\frac{3}{2}}} \quad (1)$$

$$\ddot{Y} = \frac{-(Y - Y_e)}{\left((X - X_e)^2 + (Y - Y_e)^2 + Z^2\right)^{\frac{3}{2}}} + \frac{-(Y - Y_m)}{\left((X - X_m)^2 + (Y - Y_m)^2 + Z^2\right)^{\frac{3}{2}}} \quad (2)$$

$$\ddot{Z} = \frac{-Z}{\left((X - X_e)^2 + (Y - Y_e)^2 + Z^2\right)^{\frac{3}{2}}} + \frac{-(Y - Y_m)}{\left((X - X_m)^2 + (Y - Y_m)^2 + Z^2\right)^{\frac{3}{2}}} \quad (3)$$

In which  $X$ ,  $Y$ , and  $Z$  are coordinates of the mass center of the satellite (in the inertial reference frame). In addition,  $X_e$ ,  $Y_e$ , and  $Z$  are the coordinates of center of mass of the Earth and  $X_m$ ,  $Y_m$ , and  $Z$  are the coordinates of center of mass of the Moon. Orbits of the Earth and the Moon are completely Keplerian and no perturbation is taken into account. In these equations, the Earth-Moon motion plane lies on the  $x$ - $y$  plane so the  $z$  values of the equation are set as zero for the Earth and the Moon.

By neglecting  $z$  direction of motion in the Eqs. (1-3), changing the elliptical orbit to a circular one, and then transforming inertial reference frame to rotating reference frame, equations will be reduced as follows (Henon, 1969):

$$\ddot{X} = x + 2\dot{y} - \mu' \frac{x + \mu}{r_1} - \mu \frac{x - \mu'}{r_2} \quad (4)$$

$$\ddot{y} = y - 2\dot{x} - \mu' \frac{y}{r_1} - \mu \frac{y}{r_2} \quad (5)$$

$$r_1 = ((x + \mu)^2 + y^2)^{\frac{3}{2}}, r_2 = ((x - \mu')^2 + y^2)^{\frac{3}{2}} \quad (6)$$

$$\mu' = 1 - \mu \quad (7)$$

It is noted that  $\mu$  is the ratio of mass of the Moon ( $m_m$ ) to the whole mass ( $m_e + m_m$ ).

Equations (4-7) are known as Hill's equations. In 1963, Arenstorf studied the periodic orbits of Eqs. (4-7) and proved that a family of orbits is obtained like Fig. 1.

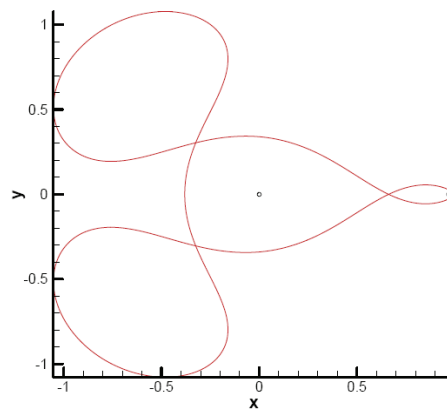


Fig. 1. Sample for Arenstorf orbit in Hill's problem

Additional works were done by Szebehely. Then in planar studies on the RTBP, Henon (1974) showed that it is practical to distinguish all orbits by only one parameter. Many works were done on the chaotic nature of RTBP. For instance, Murrison (1989) showed the Fractal nature in the Hill's equation. In his work, it was shown that an orbit with any number of oscillations could be found. In the spatial TBP, Henon (1974) showed that this family of orbits can also be addressed with only one parameter. But the major difficulty in categorizing this point is that, the starting and ending points of the orbit must both lie on the centerline of the two primaries.

In recent works, two approaches are available. One approach is based on Circular RTBP that may be expanded to more general cases. The other approach is based on studying the accurate path and measuring the control effort needed for station-keeping actions. In the first approach, there are a few studies focused on a measure of chaos like Lyapunov exponent and KS entropy. But, in the second one it is common that chaos indicators are measured for the problem.

Additional works on spatial orbits have been thoroughly studied in recent decades because of the difficulty of the problem. But since the mid 1990's some works have been done on planning new space missions using the chaotic nature of TBP. In this category, the works of Boltt (Boltt and Meiss (1995)) and Gomez et al. (2004) are among the most important ones. Gomez et al. (2004) designed a new mission based on chaotic nature of TBP. In their works, shooting to the Moon was designed using the chaotic theory. Their designed mission needed less boost effort than Hohmann orbit (standard Earth to Moon

transfer). In this category, generally, planar TBP was used and most of the works used the circular orbit for the two primaries. The works of Gomez et al. focused on reducing the cost of transfer to the Moon, leading to the longer mission. These works apply to Jupiter's moons for space missions about Jupiter. This is an applicable idea for robotic space missions and reduces the cost of a mission. In these works, it was shown that chaotic nature of TBP could be used as the base of a space mission. However, in none of them, was settling a space station in a stabilized periodic orbit around the Moon discussed. In this paper, the amount of cost and effort required for the stabilization of a space station on a periodic orbit is discussed. The periodic orbit is completely spatial and has useful points to guide the out of planet space mission.

### 3. SPATIAL ORBITS

The modeling is based on the Eqs. (1-3). In these equations, the Moon's orbit is assumed completely general. Thus, orbits with 2, 3, 4, and more revolutions around the Moon are achieved. The problem is mathematically modeled in MATLAB<sup>®</sup> and two methods are used to find the accurate orbit. In both of them, the start point of the orbit lies on the Earth-Moon centerline. The number of revolutions around the Moon is counted when the satellite passes  $xz$ -plane in the positive  $y$  direction. The  $xyz$  coordinate, along with the position of Earth- moon system is shown in Fig. 2.

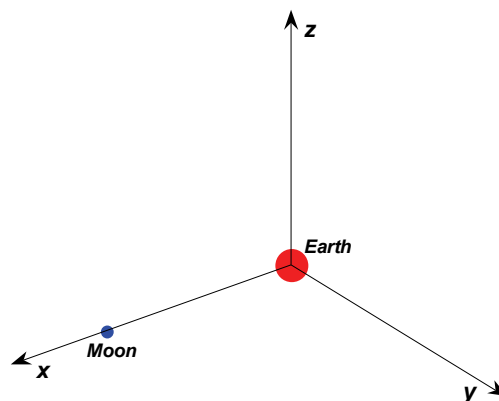


Fig. 2. xyz coordinate system

The amount of  $\mu$  (Moon to Earth mass ratio) is assumed to be 0.012277471. Henon (1974) and Bolit (1995) show that changes in the mass ratio parameter does not affect the configurations if the parameter does not exceed 0.5. Thus, the results could be extended to Sun-Earth-satellite system.

**Finding orbits:** In this section we introduce an algorithm to find the orbits. Introducing a new algorithm in a field is an important work. Farsi and Jahanmiri (2009) introduced a new algorithm in their field of study that could show the importance of new algorithms.

Because of the severity of chaos in the problem, it is assumed that the starting point is on the  $x$ -axis. In addition, since the problem geometry is symmetric with respect to the  $x$ -axis, velocity is assumed to be zero in  $x$ -direction. This assumption guarantees that the obtained orbits are closed. Although some periodic orbits may exist that never meet the  $x$ -axis, studying these orbits is more difficult. In other words, we assume that a periodic orbit meets the  $x$ -axis at least twice.

The two other components of velocity vector are found so that  $\dot{x}$  be zero in a point different from starting position. This can be done by checking the numerical solution in each time step. In other words, numerical solution continues until  $\dot{x}$  vanishes. An example of the solution from starting point to the first time  $\dot{x}$  becomes zero is shown in Fig.3.

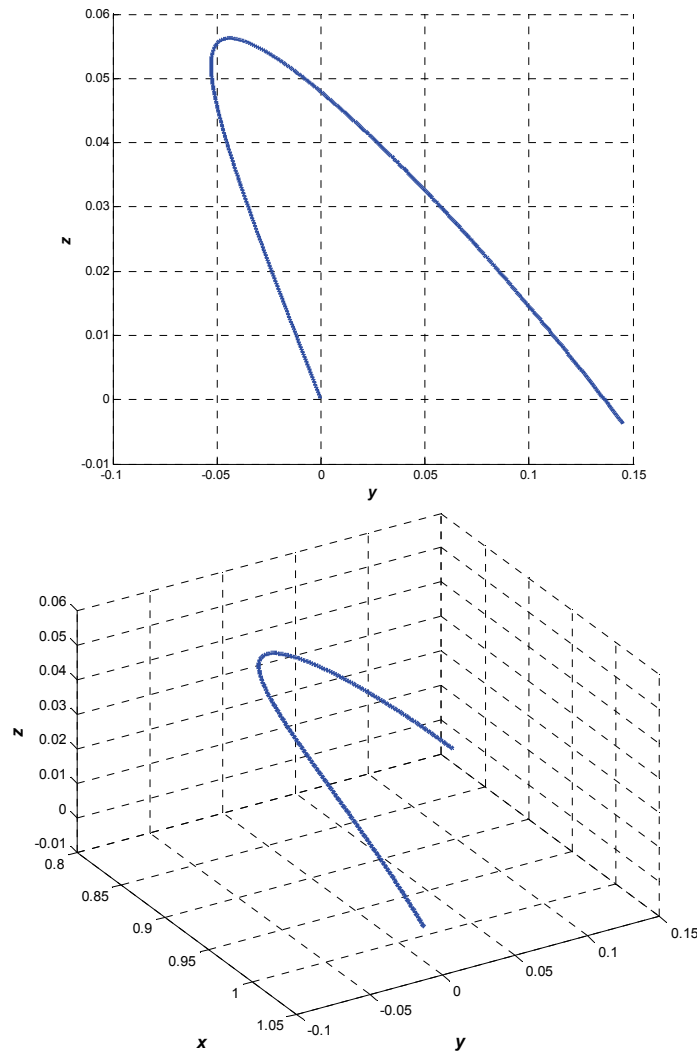


Fig. 3. Termination of integration for arbitrary values of  $\dot{y}$  and  $\dot{z}$ . It can be seen that the point of termination of integration does not lie on x-axis

The important point which must be considered is that the resulting point (in which  $\dot{x}$  is zero) may not lie on the x-axis. This condition is shown in Fig. 3.

Since the conditions for the end point mentioned above (vanishing  $y$ ,  $z$  and  $\dot{x}$ ) are not satisfied simultaneously by usual numerical methods and common optimization algorithms are not applicable (due to the chaotic nature of the problem), a new method is proposed to obtain the end point which satisfies our conditions:

At first, an arbitrary range is defined for initial values of  $\dot{y}$  and  $\dot{z}$  (for example  $[0, .9]$ ). Then, this interval is divided into smaller intervals.

Because the number of solutions is the square of number of intervals (for  $m$  intervals  $m^2$  solution processes must be done), it is better not to choose large values for division number.

After that, the numerical solution for all of the intervals is finished and the values for  $y$  and  $z$  are written into two different matrices. Now, the diagram of  $y$  and  $z$  can be sketched versus the initial values of  $\dot{y}$  and  $\dot{z}$ . This is shown in Figs. 5 and 6. It can be easily seen in Fig. 5 that the intersection of a zero height plane with the surface gives us initial values of  $\dot{y}$  which makes final  $y$  and  $z$  be zero. In a similar manner, the desired values of  $\dot{z}$  are obtained from Fig. 6.

Now, two contours are obtained, one from Fig. 5 and the other from Fig. 6. By sketching these contours in the  $\dot{y}$  -  $\dot{z}$  plane, the conjunctions are the points which make the final values of  $z$  and  $y$  be zero,

simultaneously. This is shown in Fig. 7. It is very important to note the sign of  $\dot{x}$  as it becomes zero. It can be easily seen in Fig. 4 that neglecting this point could lead to different results.

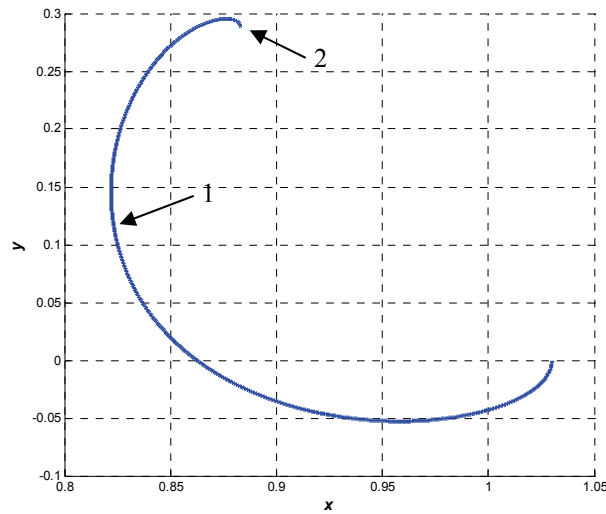


Fig. 4. Spaceship path; it is seen that if stop condition of the integration is set to “ $\dot{x} = 0$  and  $\dot{x}$  is decreasing”, the integration is terminated at Point 2. In the case when the stop condition is “ $\dot{x} = 0$  and  $\dot{x}$  is increasing”, the integration will be stopped at Point 1

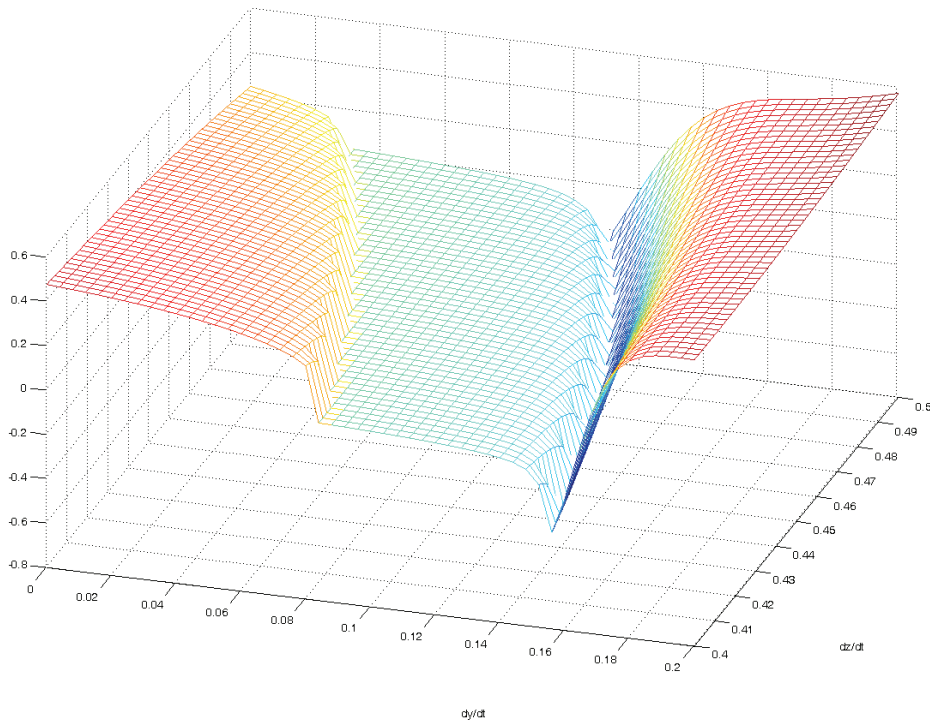


Fig. 5. Y-component of final point of the path if the initial point is  $[1.061 \ 0 \ 0]$  and the number of revolutions is 2. As we can see, the surface has no discontinuity

If the obtained points were in the intersection of two contours in smooth places, they can be trusted to be good initial guesses for finding orbits. In order to continue, an optimization program is used to minimize the distance between the resulting points and x-axis.

All the procedures mentioned above were for half of the orbit. Repeating similar steps for the next half, we can obtain the orbit completely. It is very important to note that the sign of  $\dot{x}$  must be considered as it closes to zero.

Newton-Raphson method is used to find the accurate orbit, numerically

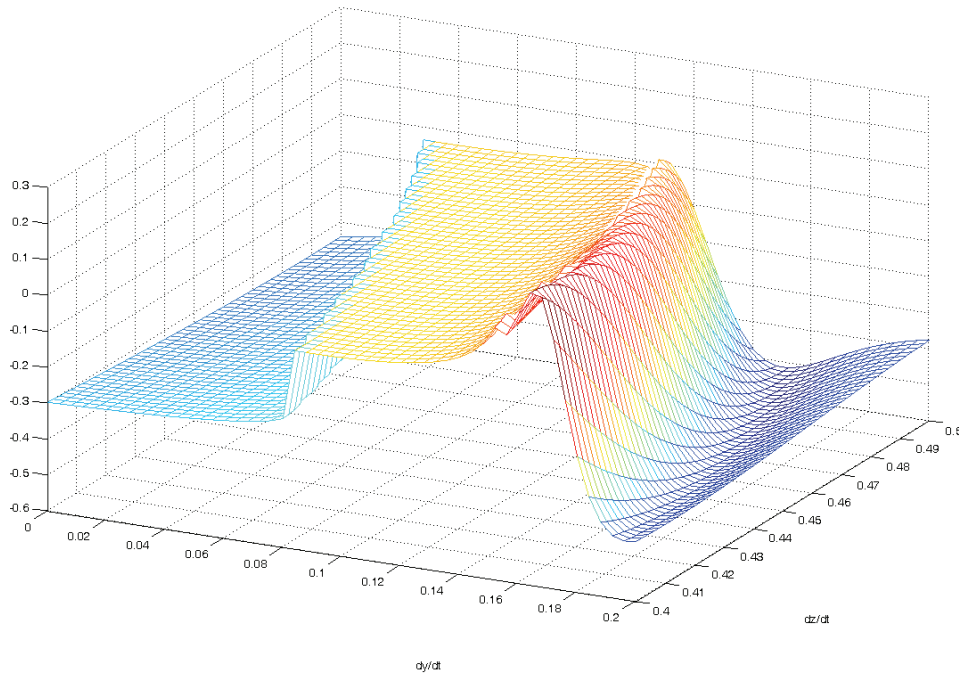


Fig. 6. Z-component of final point of the path for initial point of [1.061 0 0] and 2 revolutions; The surface has no discontinuity

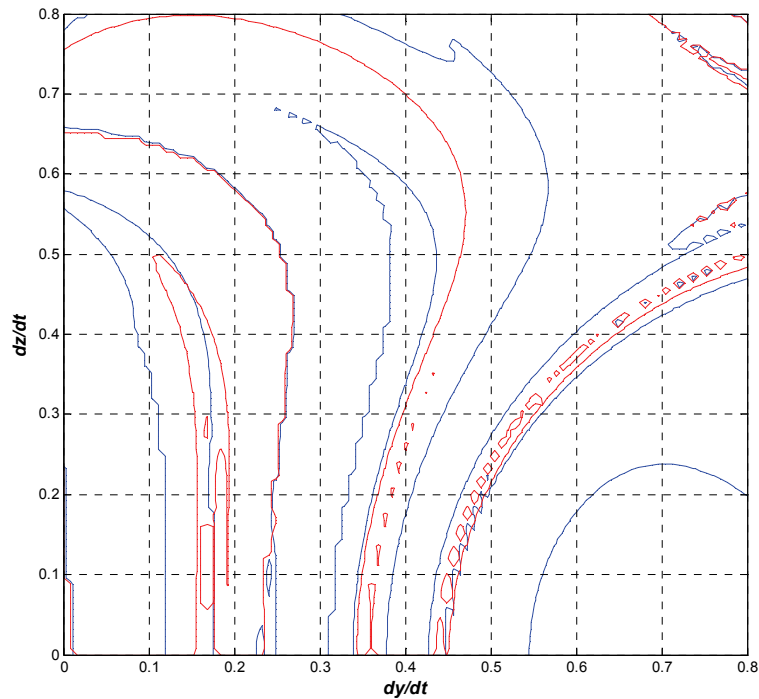


Fig. 7. Sample for finding initial guess points. Start point is on the x-axis and 2 revolutions are used for finding the orbit

After setting the number of sign changing of the velocity component  $\dot{x}$  (the integration stop condition set), Fig. 3 is achieved using the following algorithm:

*Set number of revolutions  $n$*

*Set a value for initial  $\dot{y}$*

*Set a value for initial  $\dot{z}$*

April 2012

IJST, Transactions of Mechanical Engineering, Volume 36, Number M1

*With initial  $x$ , start the integration*

*If  $\dot{x}$  equals to zero for  $n$  times, stop the integration and save  $y$  and  $z$ .*

*Next for  $\dot{z}$*

*Next for  $\dot{y}$*

*Make the surface with values of  $y$  and  $z$  versus initial  $\dot{z}$  and  $\dot{y}$*

*Plot the  $y = 0$  contour with a red line and  $z = 0$  contour with blue lines.*

*Determine cross points of red and blue lines.*

*Do refining algorithm around the cross points.*

Because of the singularity of the problem, some fault points may be determined by means of the algorithm. These points refer to non-periodic orbits. However, some of the gathered points lead to proper orbits.

Also, discontinuity in values of  $z$  or  $y$  is another reason for the two contours intersection. For example, if the sign of  $z$  values is suddenly changed, no real intersection with zero height plane occurs.

However, the method of finding the contours is numerical and a part of the  $z$  contour can be seen when the sign of  $z$  values changes. If this part intersects with a part of  $y$  contour, the intersection point leads to non-real solution and no orbit could be found using this fault point.

In other words, for finding the proper points, the values of  $z$  and  $y$  before and after the intersection must be checked. If these values are near enough to the zero, the calculations will be continued. Closeness threshold could be set 0.1 for checking whether the values are near enough or not. As mentioned previously, at the final stages of finding orbits the numerical solution was used for completing the algorithm. Initial velocity vector in this algorithm is calculated with 12 digits using double precision numbers: 12 digits are precise enough for number. That precision is equal to 1 millimeters error in position accuracy! and because the speed is in order of thousands of meters per second, 12-digit accuracy is more than the available accuracy that could be reached with rocket technology. On the other hand, station-keeping efforts usually compensate one tenth to hundredth of a meter per second in velocity error and usually solar winds and other sources of unwanted forces acting on satellite are greater than this amount of error.

If the planar model is used for finding the orbits, some orbits like Fig. 4 may be found. These planar orbits are similar to the spatial orbit in Fig. 7 that has been found by spatial model.

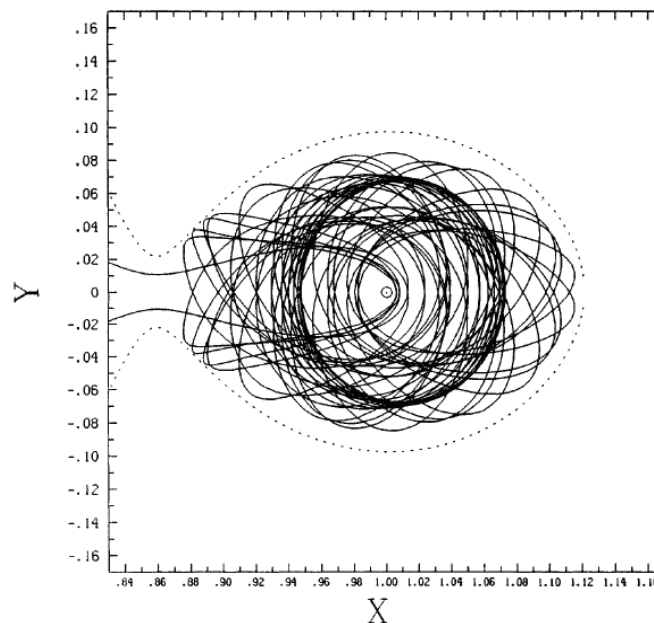


Fig. 8. The planar orbit (Murison 1989 [2]) found in a planar model is similar to the spatial orbit in Figs. 10 and 11



The refining algorithm is as follows:

For any candidate point (red and blue junction points) do the stages below:

*If the previous and the next points in both the red and blue lines are less than 0.1, continue. If not break the calculations and check another point.*

*Choose one of the four points near the junction or accurate junction point as a starting point.*

*Define the Error function as shown below*

$$\text{Error} = (\text{Error in } y \text{ direction})^2 + (\text{Error in } z \text{ direction})^2$$

*Calculate the gradient of objective function (Error function) numerically by changing the initial  $\dot{y}$  and  $\dot{z}$  a very small amount (in order of  $10^{-12}$  or less).*

*Use the Error function and gradient and Newtonian optimization method to minimize the objective function (Error function). Continue this stage till the Error function is less than threshold.*

*Save this point and try another point.*

Using this algorithm, we can simply find the orbits. It is noted that many of the candidate points are rejected in the stage 2. However, some of them lead to a proper result.

Numerical integration is done by ode45 solver in MATLAB®. The processor is an AMD Phenom II PC computer. It takes 3 days to find each orbit. Orbits with a greater number of revolutions require more time to be calculated.

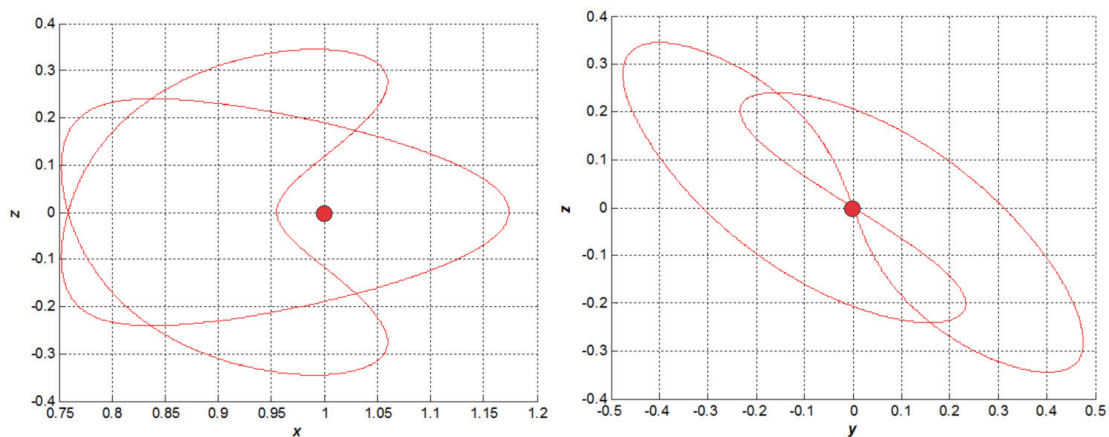


Fig. 9. Plot of 2-cycle spatial orbit around the Moon (red circle)

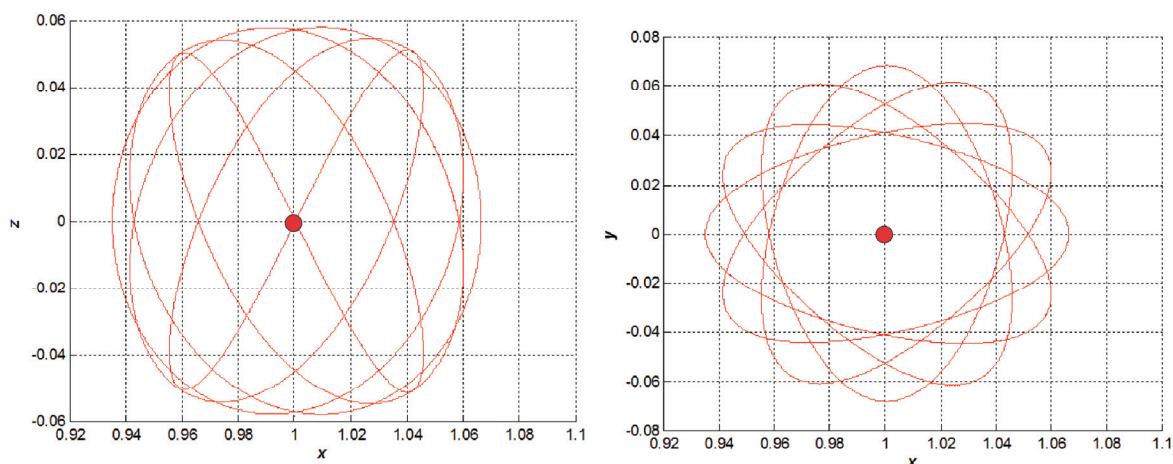


Fig. 10. Plot of 5-cycle spatial orbit around the Moon (red circle)

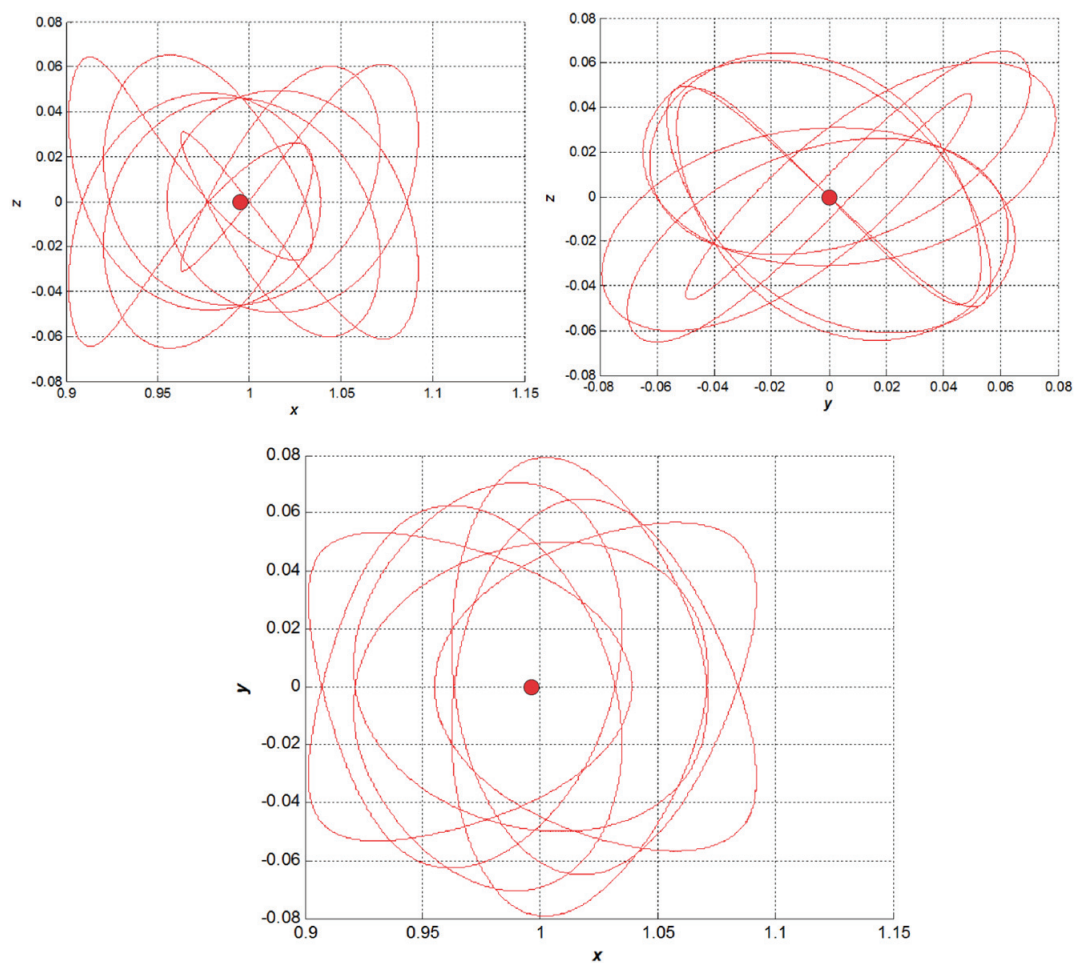


Fig. 11. Plot of 7-cycle spatial orbit around the Moon (red circle)

#### 4. CONCLUSION

Spatial orbits in TBP are much more difficult to find, solve and analyze. However, these orbits have some favorite specifications. Two new algorithms were derived for finding and refining the orbits in 3-dimensional space. These algorithms can be used for extending the planar orbits to spatial ones. The algorithms are flexible and can find the orbits in highly chaotic regions. Also, the algorithms can be used for finding the orbits in other multi-body problems.

It is necessary to mention that these orbits could be used in planning new low-cost space missions in the future decades.

#### REFERENCES

1. Szebehely, V. & Nacozy, P. (1967). A class of E. Stromgren's direct orbits in the restricted problem. *The Astronomical Journal*, Vol. 72, No. 2, pp. 184-190.
2. Henon, M. (1974). Families of periodic orbits in the three-body problem. *Celestial Mechanics*, Vol. 10, pp. 375-388.
3. Szebehely, V. (1971). Classification of the motions of three bodies in a planet. *Celestial Mechanics*, Vol. 4, No. 1, pp. 116-118.
4. Szebehely, V. (1974). Numerical investigation of a one-parameter family of triple close approaches occurring in stellar systems. *Astronomical Journal*, Vol. 79, p. 1449.

5. Canalias, E. & Masdemont, J. J. (2008). Computing natural transfers between Sun–Earth and Earth–Moon Lissajous libration point orbits. *Acta Astronautica*, Vol. Vol. 63, pp. 238–248.
6. Boltt, E. M. & Meiss, J. D. (1995). Targeting chaotic orbits to the Moon through recurrence. *Physics Letter A*, Vol. 204, pp. 373–378.
7. Gomez, G., Koon, W. S., Lo, M. W., Marsden, J. E., Masdemont, J. & Ross, S. D. (2004). Connecting orbits and invariant manifolds in the spatial restricted three-body problem. *Nonlinearity*, Vol. 17, pp. 571–1606.
8. Palacian, J. F., Yanguas, P., Fernandez, S. & Nicotra, M. (2006). Searching for periodic orbits of the spatial elliptic restricted three-body problem by double averaging. *Physica D*, Vol. 213, pp. 15–24.
9. Aram, A., Zohoor, H. & Sohrabpour, S. (2010). Spatial limit cycles around the moon in the TBP. *Acta Astronautica*, Vol. 67, No. 1-2, pp. 46–52.
10. Henon, M. (1969). Numerical exploration of the restricted problem, V. *Astron. Astrophys.*, Vol. 1, pp. 223–238.
11. Arenstorf, R. F. (1963). Periodic solutions of the restricted three body problem representing analytic continuations of Keplerian elliptic motions. *American Journal of Mathematics*, LXXXV, pp. 27–35.
12. Murison, M. A. (1989). The fractal dynamics of satellite capture in the circular restricted three-body problem. *The Astronomical Journal*, Vol. 98, No. 6, pp. 2346–2359.
13. Henon, M. (1965). Exploration numérique du problème restreint. II. Masses égales, stabilité des orbites périodiques. *Ann. Astrophysics*, Vol. 28, pp. 992–1007.
14. Farsi, M. & Jahanmiri, A. (2009). A new control algorithm for concentration control in three affect falling film evaporators. *Iranian Journal of Science and Technology Transaction-B Engineering*, Vol. 33, No. B5, pp. 387–396.