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FREE VIBRATION OF FUNCTIONALLY GRADED SIZE DEPENDENT NANOPLATES BASED ON SECOND ORDER SHEAR DEFORMATION THEORY USING NONLOCAL ELASTICITY THEORY*

M. R. NAMI AND M. JANGHORBAN**

School of Mechanical Engineering, Shiraz University, Shiraz, I. R. of Iran
Email: maziar.janghorban@gmail.com

Abstract– In this article, an analytical solution is developed to study the free vibration analysis of functionally graded rectangular nanoplates. The governing equations of motion are derived based on second order shear deformation theory using nonlocal elasticity theory. It is assumed that the material properties of nanoplate vary through the thickness according to the power law distribution. Our numerical results are compared with the results of isotropic nanoplates and functionally graded macro plates. The effects of various parameters such as nonlocal parameter and power law indexes are also investigated.

Keywords– Free vibration, nonlocal elasticity theory, second order shear deformation theory, functionally graded nanoplates

1. INTRODUCTION

Functionally graded material (FGM) may be characterized by the variation in composition and structure gradually over volume, resulting in corresponding changes in the properties of the material [1]. In past decades the free vibration of functionally graded materials has been studied extensively. Malekzadeh and Heydarpour [2] investigated the free vibration analysis of rotating functionally graded cylindrical shells subjected to thermal environment based on the first order shear deformation theory (FSDT) of shells. The formulation included the centrifugal and Coriolis forces due to rotation of the shell. The differential quadrature method was adopted to discretize the thermoelastic equilibrium equations and the equations of motion. Ungbhakorn and Wattanasakulpong [3] presented thermo-elastic vibration response of functionally graded plates carrying distributed patch mass based on third order shear deformation theory. The solutions were obtained by energy method. In addition, forced vibration analysis with external dynamic load acting on the sub-domain of the patch mass was also discussed. Kumar and Lal [4] predicted the first three natural frequencies of free axisymmetric vibration of two-directional functionally graded annular plates resting on Winkler foundation using differential quadrature method and Chebyshev collocation technique. Frequency equations for a plate clamped at both the edges and another plate simply supported at both the edges were obtained using both the methods. Based on the three-dimensional theory of elasticity and assuming that the mechanical properties of the materials vary continuously in the thickness direction and have the same exponent-law variations, the three-dimensional free and forced vibration analysis of functionally graded circular plate with various boundary conditions was achieved by Nie and Zhong [5]. Huang et al [6] investigated the free vibrations of rectangular FGM plates through internal cracks using the Ritz method. Three-dimensional elasticity theory was employed, and new sets of admissible functions for the displacement fields were proposed to enhance the effectiveness of the Ritz

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**Corresponding author

method in modeling the behaviors of cracked plates. Matsunaga [7] analyzed the natural frequencies and buckling stresses of plates made of functionally graded materials by taking into account the effects of transverse shear and normal deformations and rotatory inertia. By using the method of power series expansion of displacement components, a set of fundamental dynamic equations of a two-dimensional higher-order theory for rectangular functionally graded (FG) plates was derived through Hamilton's principle. Malekzadeh and Alibeygi Beni [8] presented the free vibration of functionally graded arbitrary straight-sided quadrilateral plates under thermal environment and based on the first order shear deformation theory. The differential quadrature method was adopted to discretize the equilibrium equations. Free vibration of functionally graded micro/nano plates was also considered in recent years. Ke et al [9] developed a non-classical microplate model for the axisymmetric nonlinear free vibration analysis of annular microplates made of functionally graded materials based on the modified couple stress theory, Mindlin plate theory and von Kármán geometric nonlinearity. The non-classical model was capable of incorporating the microplate model with the length scale parameter, geometric nonlinearity, transverse shear deformation and rotary inertia. Ke et al [10] also studied the the bending, buckling and free vibration of annular microplates made of functionally graded materials based on the modified couple stress theory and Mindlin plate theory. The material properties of the FGM microplates were assumed to vary in the thickness direction and were estimated through the Mori–Tanaka homogenization technique. Asghari and Taati [11] presented a size-dependent formulation for mechanical analyses of inhomogeneous micro-plates based on the modified couple stress theory. The governing differential equations of motion were derived for functionally graded plates with arbitrary shapes utilizing a variational approach. Utilizing the derived formulation, the free-vibration behavior as well as the static response of a rectangular FG micro-plate was proposed. Natarajan et al [12] investigated the size dependent linear free flexural vibration behavior of functionally graded nanoplates using the iso-geometric based finite element method. The field variables were approximated by non-uniform rational B-splines. The nonlocal constitutive relation was based on Eringen's differential form of nonlocal elasticity theory.

In present research, as a first endeavor, the free vibration of functionally graded nanoplates is investigated based on second order shear deformation theory using nonlocal elasticity theory. An analytical approach is used to study the free vibration of functionally graded nanoplates. It is assumed that the material properties are varying through the thickness according to power law distribution. The results of present work may be used as bench marks for future works.

2. REVIEW OF NONLOCAL ELASTICITY THEORY

Up to now, different theories have been developed with considering size effects such as nonlocal and strain gradient elasticity theories. In nonlocal theory of elasticity, the points undergo translational motion as in the classical case, but the stress at a point depends on the strain in a region near that point [13]. As for physical interpretation, the nonlocal theory incorporates long range interactions between points in a continuum model. Such long range interactions occur between charged atoms or molecules in a solid [14]. Consider a single layer graphene sheet with assumed isotropic material in continuum model. The non-local constitutive behavior of a Hookean solid can be represented by the following differential constitutive equations:

$$(1-\mu\nabla^2)\sigma = t \tag{1}$$

where μ is the nonlocal parameter and t is the macroscopic stress tensor at a point which is defined for macro structures [15-17]. As an example of studies, by considering size effects which may be potentially useful to micro/nano technology and micro/nano design and manufacturing is the bending of a mirco/nano cantilever beam, useful for the design of actuators and micro/nano probes for chemical and medical applications reported by Aifantis [18]. As another example, experimental test of the radial vibration of spherical nanoparticles made of materials with anisotropic elasticity was theoretically investigated using nonlocal continuum mechanics. The suggested model was justified by good agreement between the results given by this model and available experimental data [19].

3. GOVERNING EQUATIONS

According to the second order shear deformation theory [20-22], the displacements of an arbitrary point of the functionally graded nanoplate can be defined in terms of seven unknown parameters in Cartesian coordinate as follows:

$$\begin{aligned} u_1 &= u + zQ_1 + z^2Q_2 \\ u_2 &= v + z\psi_1 + z^2\psi_2 \\ u_3 &= w(x,y) \end{aligned} \tag{2}$$

Using the displacement form (2), the strain-displacement relations give the following strain field for second order shear deformation theory,

$$\begin{aligned} \varepsilon_{11} &= \varepsilon_{11}^0 + zK_{11} + z^2K_{11}^{'} = \frac{\partial u}{\partial x} + z\frac{\partial Q_1}{\partial x} + z^2\frac{\partial Q_2}{\partial x} \\ \varepsilon_{22} &= \varepsilon_{22}^0 + zK_{22} + z^2K_{22}^{'} = \frac{\partial v}{\partial y} + z\frac{\partial \psi_1}{\partial y} + z^2\frac{\partial \psi_2}{\partial x} \\ \gamma_{12} &= \gamma_{12}^0 + zK_{12} + z^2K_{12}^{'} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + z(\frac{\partial Q_1}{\partial y} + \frac{\partial \psi_1}{\partial x}) + z^2(\frac{\partial Q_2}{\partial y} + \frac{\partial \psi_2}{\partial x}) \\ \gamma_{23} &= \gamma_{23}^0 + z\gamma_{23}' = \psi_1 + \frac{\partial w}{\partial y} + 2z\psi_2 \\ \gamma_{13} &= \gamma_{13}^0 + z\gamma_{13}' = Q_1 + \frac{\partial w}{\partial x} + 2zQ_2 \end{aligned} \tag{3}$$

The equations of motion of the second order shear deformation theory will be derived using the dynamic version of the principle of virtual displacements [20] as follows:

$$\begin{aligned} &\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0\ddot{u} + I_2\ddot{Q}_2 + I_1Q_1 \\ &\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = I_0\ddot{v} + I_2\ddot{\psi}_2 + I_1\ddot{\psi}_1 \\ &\frac{\partial \phi_{xz}}{\partial x} + \frac{\partial \phi_{yz}}{\partial y} = I_0\ddot{w} \\ &\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_{xz} = I_2\ddot{Q}_1 + I_1\ddot{u} + I_3\ddot{Q}_2 \\ &\frac{\partial L_x}{\partial x} + \frac{\partial L_{xy}}{\partial y} - 2R_{xz} = I_2\ddot{u} + I_4\ddot{Q}_2 + I_3\ddot{Q}_1 \\ &\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_{yz} = I_2\ddot{\psi}_1 + I_1\ddot{v} + I_3\ddot{\psi}_2 \\ &\frac{\partial L_y}{\partial y} + \frac{\partial L_{xy}}{\partial x} - 2R_{yz} = I_2\ddot{v} + I_4\ddot{\psi}_2 + I_3\ddot{\psi}_1 \end{aligned} \tag{4}$$

where

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} dz \qquad \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} zdz$$

$$\begin{Bmatrix} L_x \\ L_y \\ L_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} z^2 dz \qquad \begin{Bmatrix} \phi_{xz} \\ \phi_{yz} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} dz$$

$$\begin{Bmatrix} Q_{xz} \\ Q_{yz} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} zdz$$

and $I_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho_0 (z)^i dz \ \ (\mathrm{i} = 0,...,6)$.

On the other hand, it has been previously pointed out that for the free vibration of functionally graded nanoplates, the size effects should be considered. So the stress-strain relations can be defined as [23, 24],

$$\begin{Bmatrix} \sigma_x - \mu\nabla^2(\sigma_x) \\ \sigma_y - \mu\nabla^2(\sigma_y) \\ \tau_{yz} - \mu\nabla^2(\tau_{yz}) \\ \tau_{xz} - \mu\nabla^2(\tau_{xz}) \\ \tau_{xy} - \mu\nabla^2(\tau_{xy}) \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} \tag{5a-e}$$

where μ is the nonlocal parameter and the elastic constants for functionally graded nanoplate can be expressed as,

$$Q_{11} = Q_{22} = \frac{E(z)}{1-v^2} \qquad Q_{12} = \frac{E(z)v}{1-v^2} \qquad Q_{44} = Q_{55} = Q_{66} = \frac{E(z)}{2(1+v)} \tag{6}$$

where E and v are the Young's modulus and Poisson's ratio, respectively. Integrating these results yields the stress and moment resultants as follows:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} (equation5(a))dz$$
$$N_x - \mu\nabla^2(N_x) = A_{11}\frac{\partial u}{\partial x} + B_{11}\frac{\partial Q_1}{\partial x} + D_{11}\frac{\partial Q_2}{\partial x} + A_{12}\frac{\partial v}{\partial y} + B_{12}\frac{\partial \psi_1}{\partial y} + D_{12}\frac{\partial \psi_2}{\partial y} \tag{7}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} (equation5(b))dz$$
$$N_y - \mu\nabla^2(N_y) = A_{12}\frac{\partial u}{\partial x} + B_{12}\frac{\partial Q_1}{\partial x} + D_{12}\frac{\partial Q_2}{\partial x} + A_{22}\frac{\partial v}{\partial y} + B_{22}\frac{\partial \psi_1}{\partial y} + D_{22}\frac{\partial \psi_2}{\partial y} \tag{8}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} (equation5(e))dz$$
$$N_{xy} - \mu\nabla^2(N_{xy}) = A_{66}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) + B_{66}(\frac{\partial Q_1}{\partial y} + \frac{\partial \psi_1}{\partial x}) + D_{66}(\frac{\partial Q_2}{\partial y} + \frac{\partial \psi_2}{\partial x}) \tag{9}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}}(equation5(a))zdz$$
$$M_x-\mu\nabla^2(M_x)=B_{11}\frac{\partial u}{\partial x}+D_{11}\frac{\partial Q_1}{\partial x}+E_{11}\frac{\partial Q_2}{\partial x}+B_{12}\frac{\partial v}{\partial y}+D_{12}\frac{\partial \psi_1}{\partial y}+E_{12}\frac{\partial \psi_2}{\partial y} \tag{10}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}}(equation5(b))zdz$$
$$M_y-\mu\nabla^2(M_y)=B_{12}\frac{\partial u}{\partial x}+D_{12}\frac{\partial Q_1}{\partial x}+E_{12}\frac{\partial Q_2}{\partial x}+B_{22}\frac{\partial v}{\partial y}+D_{22}\frac{\partial \psi_1}{\partial y}+E_{22}\frac{\partial \psi_2}{\partial y} \tag{11}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}}(equation5(e))zdz$$
$$M_{xy}-\mu\nabla^2(M_{xy})=B_{66}(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x})+D_{66}(\frac{\partial Q_1}{\partial y}+\frac{\partial \psi_1}{\partial x})+E_{66}(\frac{\partial Q_2}{\partial y}+\frac{\partial \psi_2}{\partial x}) \tag{12}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}}(equation5(a))z^2dz$$
$$L_x-\mu\nabla^2(L_x)=D_{11}\frac{\partial u}{\partial x}+E_{11}\frac{\partial Q_1}{\partial x}+F_{11}\frac{\partial Q_2}{\partial x}+D_{12}\frac{\partial v}{\partial y}+E_{12}\frac{\partial \psi_1}{\partial y}+F_{12}\frac{\partial \psi_2}{\partial y} \tag{13}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}}(equation5(b))z^2dz$$
$$L_y-\mu\nabla^2(L_y)=D_{12}\frac{\partial u}{\partial x}+E_{12}\frac{\partial Q_1}{\partial x}+F_{12}\frac{\partial Q_2}{\partial x}+D_{22}\frac{\partial v}{\partial y}+E_{22}\frac{\partial \psi_1}{\partial y}+F_{22}\frac{\partial \psi_2}{\partial y} \tag{14}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}}(equation5(e))z^2dz$$
$$L_{xy}-\mu\nabla^2(L_{xy})=D_{66}(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x})+E_{66}(\frac{\partial Q_1}{\partial y}+\frac{\partial \psi_1}{\partial x})+F_{66}(\frac{\partial Q_2}{\partial y}+\frac{\partial \psi_2}{\partial x}) \tag{15}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}}(equation5(d))dz$$
$$Q_{xz}-\mu\nabla^2(Q_{xz})=A_{55}(Q_1+\frac{\partial w}{\partial x})+B_{55}(2Q_2) \tag{16}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}}(equation5(c))dz$$
$$Q_{yz}-\mu\nabla^2(Q_{yz})=A_{44}(\psi_1+\frac{\partial w}{\partial y})+B_{55}(2\psi_2) \tag{17}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}}(equation5(d))zdz$$
$$R_{xz}-\mu\nabla^2(R_{xz})=B_{55}(Q_1+\frac{\partial w}{\partial x})+D_{55}(2Q_2) \tag{18}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}}(equation5(c))zdz$$
$$R_{yz}-\mu\nabla^2(R_{yz})=B_{44}(\psi_1+\frac{\partial w}{\partial y})+D_{55}(2\psi_2) \tag{19}$$

Where

$$A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{ij}(1, z, z^2, z^3, z^4)dz$$

Now, by combining appropriate equations by taking seven equilibrium Eqs. (4) into consideration, one can easily obtain the governing differential equations for functionally graded nanoplate. These equations are expressed as,

$$\frac{\partial}{\partial x}(equation(7) + \frac{\partial}{\partial y}(equation(9)):$$

$$(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y}) - \mu\nabla^2(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y}) = A_{11}\frac{\partial^2 u}{\partial x^2} + B_{11}\frac{\partial^2 Q_1}{\partial x^2} + D_{11}\frac{\partial^2 Q_2}{\partial x^2} + A_{12}\frac{\partial^2 V}{\partial x \partial y} \tag{20}$$

$$+ B_{12}\frac{\partial^2 \psi_1}{\partial x \partial y} + D_{12}\frac{\partial^2 \psi_2}{\partial x \partial y} + A_{66}(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}) + B_{66}(\frac{\partial^2 Q_1}{\partial y^2} + \frac{\partial^2 \psi_1}{\partial x \partial y}) + D_{66}(\frac{\partial^2 Q_2}{\partial y^2} + \frac{\partial^2 \psi_2}{\partial x \partial y})$$

$$\frac{\partial}{\partial y}(equation(8) + \frac{\partial}{\partial x}(equation(9)):$$

$$(\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x}) - \mu\nabla^2(\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x}) = A_{12}\frac{\partial^2 u}{\partial x \partial y} + B_{12}\frac{\partial^2 Q_1}{\partial x \partial y} + D_{12}\frac{\partial^2 Q_2}{\partial x \partial y} + A_{22}\frac{\partial^2 v}{\partial y^2} \tag{21}$$

$$+ B_{22}\frac{\partial^2 \psi_1}{\partial y^2} + D_{22}\frac{\partial^2 \psi_2}{\partial y^2} + A_{66}(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2}) + B_{66}(\frac{\partial^2 Q_1}{\partial x \partial y} + \frac{\partial^2 \psi_1}{\partial x^2}) + D_{66}(\frac{\partial^2 Q_2}{\partial x \partial y} + \frac{\partial^2 \psi_2}{\partial x^2})$$

$$\frac{\partial}{\partial x}(equation(16) + \frac{\partial}{\partial y}(equation(17)):$$

$$(\frac{\partial Q_{xz}}{\partial x} + \frac{\partial Q_{yz}}{\partial y}) - \mu\nabla^2(\frac{\partial Q_{xz}}{\partial x} + \frac{\partial Q_{yz}}{\partial y}) = A_{55}(\frac{\partial Q_1}{\partial x} + \frac{\partial^2 w}{\partial x^2}) + B_{55}(2\frac{\partial Q_2}{\partial x}) + A_{44}(\frac{\partial \psi_1}{\partial y} + \frac{\partial^2 w}{\partial y^2}) + B_{44}(2\frac{\partial \psi_2}{\partial y}) \tag{22}$$

$$\frac{\partial}{\partial x}(equation(10) + \frac{\partial}{\partial y}(equation(12) - (equation(16)):$$

$$(\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_{xz}) - \mu\nabla^2(\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_{xz}) = B_{11}\frac{\partial^2 u}{\partial x^2} + D_{11}\frac{\partial^2 Q_1}{\partial x^2} + E_{11}\frac{\partial^2 Q_2}{\partial x^2} + B_{12}\frac{\partial^2 v}{\partial x \partial y}$$

$$+ D_{12}\frac{\partial^2 \psi_1}{\partial x \partial y} + E_{12}\frac{\partial^2 \psi_2}{\partial x \partial y} + B_{66}(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}) + D_{66}(\frac{\partial^2 Q_1}{\partial y^2} + \frac{\partial^2 \psi_1}{\partial x \partial y}) + E_{66}(\frac{\partial^2 Q_2}{\partial y^2} + \frac{\partial^2 \psi_2}{\partial x \partial y}) \tag{23}$$

$$- A_{55}(Q_1 + \frac{\partial W}{\partial x}) - B_{55}(2Q_2)$$

$$\frac{\partial}{\partial y}(equation(11) + \frac{\partial}{\partial x}(equation(12) - (equation(17)):$$

$$(\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_{yz}) - \mu\nabla^2(\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_{yz}) = B_{12}\frac{\partial^2 u}{\partial x \partial y} + D_{12}\frac{\partial^2 Q_1}{\partial x \partial y} + E_{12}\frac{\partial^2 Q_2}{\partial x \partial y}$$

$$+ B_{22}\frac{\partial^2 V}{\partial y^2} + D_{22}\frac{\partial^2 \psi_1}{\partial y^2} + E_{22}\frac{\partial^2 \psi_2}{\partial y^2} + B_{66}(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2}) + D_{66}(\frac{\partial^2 Q_1}{\partial x \partial y} + \frac{\partial^2 \psi_1}{\partial x^2}) + E_{66}(\frac{\partial^2 Q_2}{\partial x \partial y} + \frac{\partial^2 \psi_2}{\partial x^2}) \tag{24}$$

$$- A_{44}(\psi_1 + \frac{\partial w}{\partial y}) - B_{44}(2\psi_2)$$

$$\frac{\partial}{\partial x}(equation(13)+\frac{\partial}{\partial y}(equation(15)-2(equation(18)):$$

$$\begin{aligned}&(\frac{\partial L_x}{\partial x}+\frac{\partial L_{xy}}{\partial y}-2R_{xz})-\mu\nabla^2(\frac{\partial L_x}{\partial x}+\frac{\partial L_{xy}}{\partial y}-2R_{xz})=D_{11}\frac{\partial^2 u}{\partial x^2}+E_{11}\frac{\partial^2 Q_1}{\partial x^2}+F_{11}\frac{\partial^2 Q_2}{\partial x^2}+D_{12}\frac{\partial^2 v}{\partial x\partial y}\\&+E_{12}\frac{\partial^2\psi_1}{\partial x\partial y}+F_{12}\frac{\partial^2\psi_2}{\partial x\partial y}+D_{66}(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\partial y})+E_{66}(\frac{\partial^2 Q_1}{\partial y^2}+\frac{\partial^2\psi_1}{\partial x\partial y})+F_{66}(\frac{\partial^2 Q_2}{\partial y^2}+\frac{\partial^2\psi_2}{\partial x\partial y})\\&-2B_{55}(Q_1+\frac{\partial W}{\partial x})-2D_{55}(2Q_2)\end{aligned}\tag{25}$$

$$\frac{\partial}{\partial y}(equation(14)+\frac{\partial}{\partial x}(equation(15)-2(equation(19)):$$

$$\begin{aligned}&(\frac{\partial L_y}{\partial y}+\frac{\partial L_{xy}}{\partial x}-2R_{yz})-\mu\nabla^2(\frac{\partial L_y}{\partial y}+\frac{\partial L_{xy}}{\partial x}-2R_{yz})=D_{12}\frac{\partial^2 u}{\partial x\partial y}+E_{12}\frac{\partial^2 Q_1}{\partial x\partial y}+F_{12}\frac{\partial^2 Q_2}{\partial x\partial y}\\&+D_{22}\frac{\partial^2 v}{\partial y^2}+E_{22}\frac{\partial^2\psi_1}{\partial y^2}+F_{22}\frac{\partial^2\psi_2}{\partial y^2}+D_{66}(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x^2})+E_{66}(\frac{\partial^2 Q_1}{\partial x\partial y}+\frac{\partial^2\psi_1}{\partial x^2})+F_{66}(\frac{\partial^2 Q_2}{\partial x\partial y}+\frac{\partial^2\psi_2}{\partial x^2})\\&-2B_{44}(\psi_1+\frac{\partial w}{\partial y})-2D_{44}(2\psi_2)\end{aligned}\tag{26}$$

This completes the development of second order shear deformation theory for studying the free vibration of rectangular functionally graded nanoplates. It is mentioned that the equations of motion of the first order plate theory and Kirchhoff plate theory can be achieved from the above equations for isotropic and functionally graded nanoplates. In fact, the above governing differential equations on the basis of second order theory are similar to those for first order theory for macro plates [20]. Due to the fact that the nonlocal parameter loses its effect at the edges of the plate in view of deflections being zero there, simply supported boundary conditions for the nonlocal plate are the same as those of the local plate theory [25]. So the Navier solution may be a good assumption for solving the above equations. In the present work, it is assumed that,

$$\begin{aligned}u&=\sum\sum u\cos\alpha x\sin\beta ye^{+i\omega t}\\v&=\sum\sum v\sin\alpha x\cos\beta ye^{+i\omega t}\\w&=\sum\sum w\sin\alpha x\sin\beta ye^{+i\omega t}\\Q_1&=\sum\sum Q_1\cos\alpha x\sin\beta ye^{+i\omega t}\\Q_2&=\sum\sum Q_2\cos\alpha x\sin\beta ye^{+i\omega t}\\\psi_1&=\sum\sum\psi_1\sin\alpha x\cos\beta ye^{+i\omega t}\\\psi_2&=\sum\sum\psi_2\sin\alpha x\cos\beta ye^{+i\omega t}\end{aligned}$$

Thus, the governing equations for functionally graded nanoplates are satisfied for simply supported boundary condition. To compute the natural frequencies, it is better to rewrite the above equations in the following form,

$$([K]+\omega^2[M])\boldsymbol{X}=0\tag{27}$$

Where $\boldsymbol{X}=[\boldsymbol{u}\ \boldsymbol{v}\ \boldsymbol{w}\ Q_1\ \ Q_2\ \psi_1\ \ \psi_2]^{\boldsymbol{T}}$**.** The components of matrices in Eq. (27) are defined in the appendix.

4. NUMERICAL RESULTS

In this section, first of all, the accuracy of the present formulation is studied through examples of isotropic nanoplate and functionally graded macro plate. Then, a parametric study is carried out to show the influences of different parameters such as nonlocal parameter and power law index. It is important to mention that the material properties of functionally graded are assumed as,

$$E = (E_c - E_m)(\frac{z}{h} + 0.5)^p + E_m$$
$$\rho = (\rho_c - \rho_m)(\frac{z}{h} + 0.5)^p + \rho_m \tag{28}$$

where $E_c = 380GPa, \rho_c = 3800,\ E_m = 70GPa,\ \rho_m = 2702$ [16]. In Table 1, current results are compared with the results of first order shear deformation theory [16] and higher order shear deformation theory [16] for functionally graded macro plates. It can be seen that for different power law indexes, the non-dimensional frequencies are in good agreement, especially with the results of higher order shear deformation theory. In this example, the non-dimensional frequency is defined as,

$$\boldsymbol{\omega} = \omega h \sqrt{\frac{E_c}{\rho_c}} \tag{29}$$

In Table 2, the present nonlocal second order shear deformation theory is compared with nonlocal first order shear deformation theory. It is noted that the nanoplate in this example is isotropic. It is shown that for different modes of vibration, the results are in good agreement. In this table the frequency ratio is considered as follows:

$$\text{Frequency ratio} = \frac{\text{Natural frequency using nonlocal theory}}{\text{Natural frequency using local theory}}$$

It is shown that with the increase of nonlocal parameter, the frequency ratio decreases for all modes of vibration. Figure 1 depicts the effects of power law indexes for different modes. In this figure, the nonlocal parameter is assumed to be 0.04 nm^2. It can be seen that increasing the power law index will cause the non-dimensional frequencies to decrease. It is also shown that the above result is independent of the mode of vibration. Figure 2 illustrates the effects of both length to thickness ratio and power law index on the non-dimensional frequencies of FG nanoplates. The value of nonlocal parameter is the same as Table 1. According to Table 1 and Fig. 2, it is found that with the increase of power law indexes, the frequencies decrease for functionally graded macro and nano plates. Moreover, one can easily see that increasing the length to thickness ratio will decrease the natural frequencies. In Fig. 3, the influences of both nonlocal parameter and power law index are presented for simply supported FG nanoplates. In this figure the non-dimensional parameter g is defined as,

$$g = \frac{\sqrt{\mu}}{a} \tag{30}$$

where a is the length of nanoplate. It is shown that with increasing the nonlocal parameter and power law index, the natural frequencies will decrease. The same result is found for mode (2,2) in Fig. 4. From Figs. 3 and 4, it is also found that by increasing the nonlocal parameter, the rate of variation of non-dimensional frequencies will decrease. From these figures it is shown that in investigating the FG nanoplates, the effects of nonlocal parameter cannot be ignored so the theories for macro plates are not suitable for nanoplates. Figure 5 shows the influences of both nonlocal parameter and power law index on frequency ratios. One can easily find that increasing the parameter g will cause the frequency ratios to decrease but the power law indexes do not have a special effect on the frequency ratios. It may be important to note that the behaviors of non-dimensional frequency and frequency ratio are not the same and they can be studied separately.

Table 1. Non-dimensional frequencies of functionally graded square macro plate ($\mu = 0, \frac{a}{h} = 20$)

p	Present	HSDT[16]	FSDT[16]
0	0.0148	0.0148	0.0146
1	0.0113	0.0113	0.0113
4	0.0098	0.0098	0.0098
10	0.0094	0.0094	0.0094

Table 2. Frequency ratios for isotropic square nanoplate

		$\sqrt{\mu}$		
(1,1)	0.0	0.2	0.4	0.6
Present	1	0.7475	0.4904	0.3512
FSDT[17]	1	0.7475	0.4904	0.3512
(1,2)				
Present	1	0.5799	0.3353	0.2308
FSDT[17]	1	0.5799	0.3353	0.2308
(2,2)				
Present	1	0.4904	0.2708	0.1844
FSDT[17]	1	0.4904	0.2708	0.1844

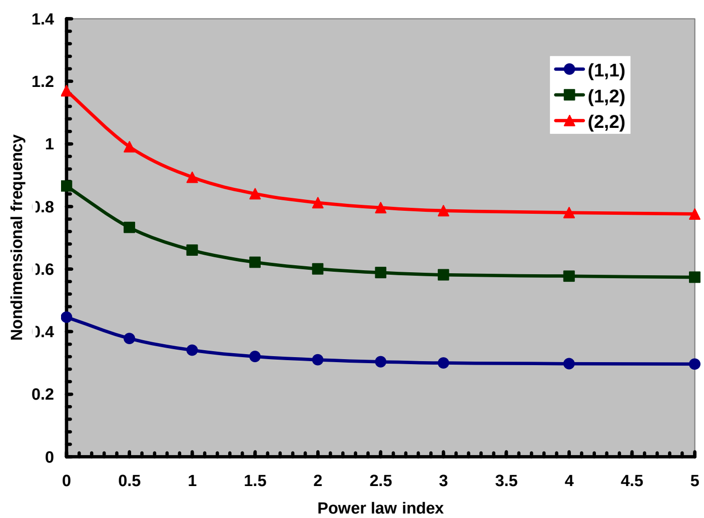

Fig 1. The effects of power law index on non-dimensional frequencies ($\omega = \omega \times 10^3$) for different modes

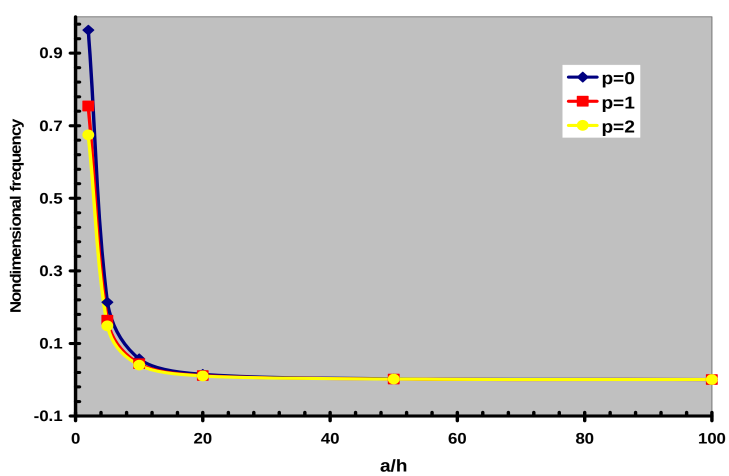

Fig 2. The effects of length to thickness ratio and power law index on non-dimensional frequencies

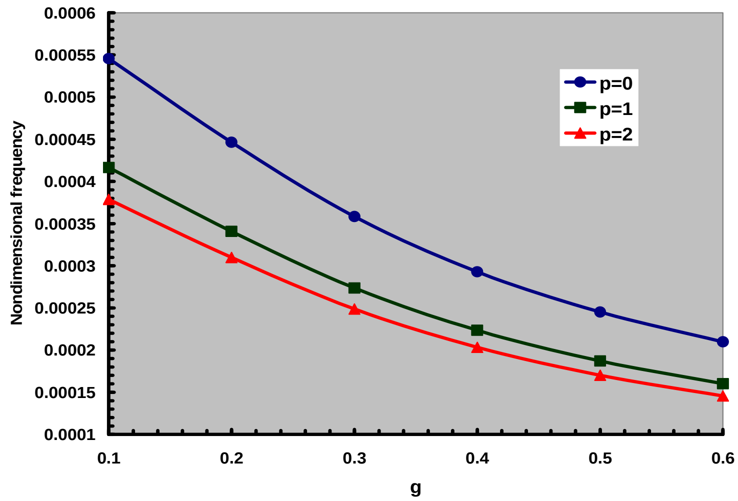

Fig 3. The effects of parameter g and power law index on non-dimensional frequencies (mode (1,1))

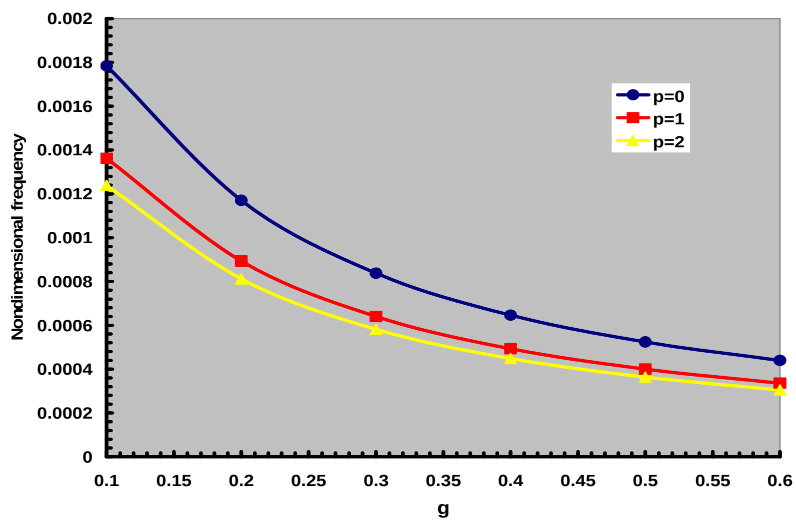

Fig 4. The effects of parameter g and power law index on non-dimensional frequencies (mode (2,2))

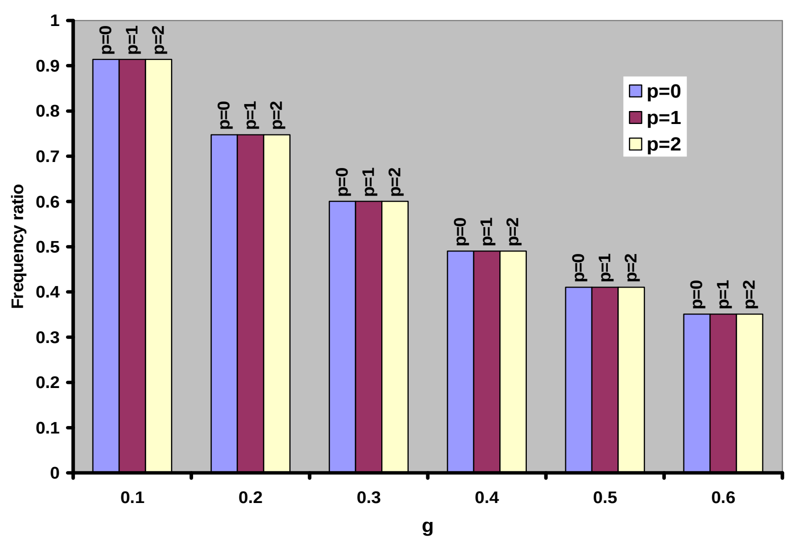

Fig 5. The effects of parameter g and power law index on frequency ratios

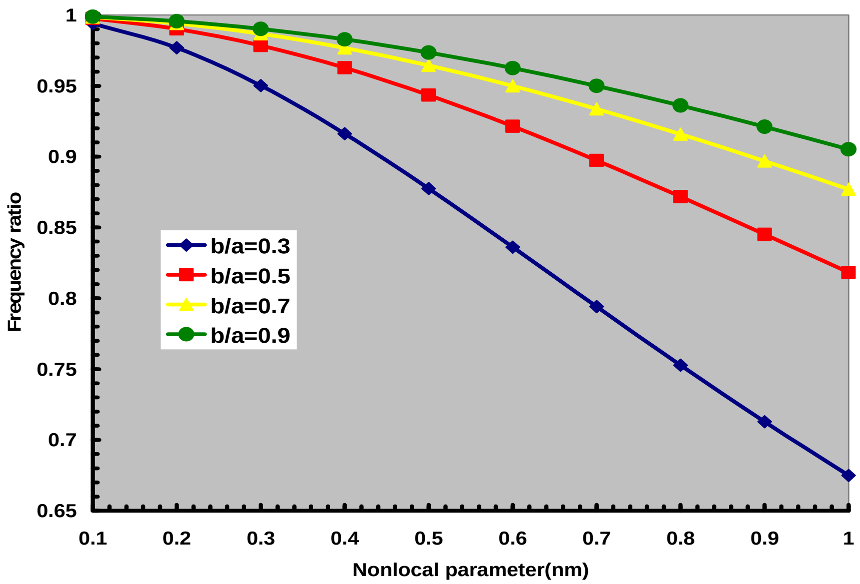

Fig 6. The effects of aspect ratio and nonlocal parameter on frequency ratios (1,1)

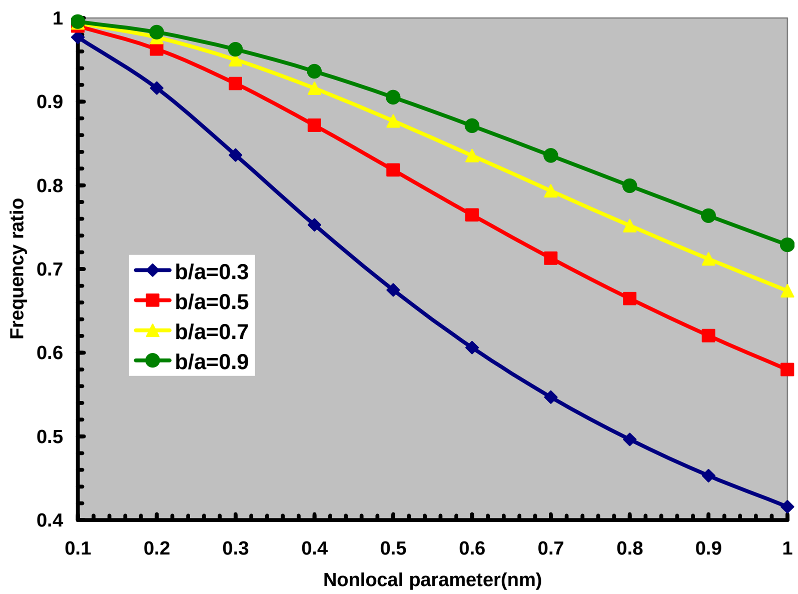

Fig 7. The effects of aspect ratio and nonlocal parameter on frequency ratios (2,2)

In Figs. 6 and 7, the effects of aspect ratio and nonlocal parameter on the frequency ratios of rectangular nanoplates are shown for different modes of vibration. It is shown that with the increase of aspect ratio, the frequency ratios increase. It is illustrated that for lower aspect ratios, the influence of nonlocal parameters decreases. From these figures, it seems that the frequency ratios for mode (2,2) are less than those for mode (1,1). Finally, it is noted that the present methodology can be used for investigating other nano structures, too [26, 27].

5. CONCLUSION

A Navier method was applied to the free vibration of functionally graded rectangular nanoplates. The formulations were based on second order shear deformation theory using nonlocal elasticity theory. The results from this method agreed with those of FSDT and HSDT for FG macro plates. The present results were also in good agreement for isotropic nanoplates based on nonlocal FSDT. It was shown that,

- Increasing the power law index will cause the non-dimensional frequencies to decrease.
- Increasing the length to thickness ratio will decrease the natural frequencies.
- Increasing the parameter g will cause the frequency ratios to decrease.

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APPENDIX A

The components of matrices in equation (27) are defined as follow,

$K_{11} = -\alpha^2 A_{11} - \beta^2 A_{66}$

$K_{12} = -\alpha\beta A_{12} - \alpha\beta A_{66}$

$K_{14} = -\alpha^2 B_{11} - \beta^2 B_{66}$

$K_{15} = -\alpha^2 D_{11} - \beta^2 D_{66}$

$K_{16} = -\alpha\beta B_{12} - \alpha\beta B_{66}$

$K_{17} = -\alpha\beta D_{12} - \alpha\beta D_{66}$

$K_{21} = -\alpha\beta A_{12} - \alpha\beta A_{66}$

$K_{22} = -\beta^2 A_{22} - \alpha^2 A_{66}$

$K_{24} = -\alpha\beta B_{12} - \alpha\beta B_{66}$

$K_{25} = -\alpha\beta D_{12} - \alpha\beta B_{66}$

$K_{26} = -\beta^2 B_{22} - \alpha^2 B_{66}$

$K_{27} = -\beta^2 D_{22} - \alpha^2 D_{66}$

$K_{33} = -\alpha^2 A_{55} - \beta^2 A_{44}$

$K_{34} = -\alpha A_{55}$

$K_{35} = -2\alpha\beta_{55}$

$K_{36} = -\beta A_{44}$

$K_{37} = -2\beta B_{44}$

$K_{41} = -\alpha^2 B_{11} - \beta^2 B_{66}$

$K_{62} = -\alpha\beta D_{12} - \alpha\beta D_{66}$

$K_{63} = -2\alpha B_{55}$

$K_{64} = -\alpha^2 E_{11} - \beta^2 E_{66} - 2B_{55}$

$K_{65} = -\alpha^2 F_{11} - \beta^2 F_{66} - 4D_{55}$

$K_{66} = -\alpha\beta E_{12} - \alpha\beta E_{66}$

$K_{67} = -\alpha\beta F_{12} - \alpha\beta F_{66}$

$K_{71} = -\alpha\beta D_{12} - \alpha\beta D_{66}$

$K_{72} = -\beta^2 D_{22} - \alpha^2 D_{66}$

$K_{73} = -2\beta B_{44}$

$K_{74} = -\alpha\beta E_{12} - \alpha\beta E_{66}$

$K_{75} = -\alpha\beta F_{12} - \alpha\beta F_{66}$

$K_{76} = -\beta^2 E_{22} - \alpha^2 E_{66} - 2B_{44}$

$K_{77} = -\beta^2 F_{22} - \alpha^2 F_{66} - 4D_{44}$

$M_{11} = (I_0) - \mu(I_0)(-\alpha^2 - \beta^2)$

$M_{14} = (I_1) - \mu(I_1)(-\alpha^2 - \beta^2)$

$M_{15} = (I_2) - \mu(I_2)(-\alpha^2 - \beta^2)$

$M_{22} = (I_0) - \mu(I_0)(-\alpha^2 - \beta^2)$

$M_{26} = (I_1) - \mu(I_1)(-\alpha^2 - \beta^2)$

$K_{42} = -\alpha\beta B_{12} - \alpha\beta B_{66}$

$K_{43} = -\alpha A_{55}$

$K_{44} = -\alpha^2 D_{11} - B^2 D_{66} - A_{55}$

$K_{45} = -\alpha^2 E_{11} - \beta^2 E_{66} - 2B_{55}$

$K_{46} = -\alpha\beta D_{12} - \alpha\beta D_{66}$

$K_{47} = -\alpha\beta E_{12} - \alpha\beta E_{66}$

$K_{51} = -\alpha\beta B_{12} - \alpha\beta B_{66}$

$K_{52} = -\beta^2 B_{22} - \alpha^2 B_{66}$

$K_{53} = -\beta A_{44}$

$K_{54} = -\alpha\beta D_{12} - \alpha\beta D_{66}$

$K_{55} = -\alpha\beta E_{12} - \alpha\beta E_{66}$

$K_{56} = -\beta^2 D_{22} - \alpha^2 D_{66} - A_{44}$

$K_{57} = -\beta^2 E_{22} - \alpha^2 E_{66} - 2B_{44}$

$K_{61} = -\alpha^2 D_{11} - \beta^2 D_{66}$

$M_{27} = (I_2) - \mu(I_2)(-\alpha^2 - \beta^2)$

$M_{33} = (I_0) - \mu(I_0)(-\alpha^2 - \beta^2)$

$M_{41} = (I_1) - \mu(I_1)(-\alpha^2 - \beta^2)$

$M_{44} = (I_2) - \mu(I_2)(-\alpha^2 - \beta^2)$

$M_{45} = (I_3) - \mu(I_3)(-\alpha^2 - \beta^2)$

$M_{52} = (I_1) - \mu(I_1)(-\alpha^2 - \beta^2)$

$M_{56} = (I_2) - \mu(I_2)(-\alpha^2 - \beta^2)$

$M_{57} = (I_3) - \mu(I_3)(-\alpha^2 - \beta^2)$

$M_{61} = (I_2) - \mu(I_2)(-\alpha^2 - \beta^2)$

$M_{64} = (I_3) - \mu(I_3)(-\alpha^2 - \beta^2)$

$M_{65} = (I_4) - \mu(I_4)(-\alpha^2 - \beta^2)$

$M_{72} = (I_2) - \mu(I_2)(-\alpha^2 - \beta^2)$

$M_{76} = (I_3) - \mu(I_3)(-\alpha^2 - \beta^2)$

$M_{77} = (I_4) - \mu(I_4)(-\alpha^2 - \beta^2)$